ON THE POISSON VARIATE WITH BETA PRIOR DISTRIBUTION AND A NUMERICAL METHOD IN ESTIMATING THE RESULTING UNCONDITIONAL DISTRIBUTION

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THESIS

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ABSTRACT

A discrete distribution arises from a Poisson distribution with parameter λ, when the distribution of λ itself is of the form $c\lambda^{\alpha-1}(b-\lambda)^{\beta-1}$ $\lambda\varepsilon[0,b]$, where c is a scaling factor and α,β,b are strictly positive parameters. However, the functional form of the resulting unconditional distribution is not particularly tractable, hence the study of the statistical properties of the unconditional distribution is limited in the study of its mean and its variance.

In regards to the modelling of a real situation, an estimation procedure of the parameters involved in $c\lambda^{\alpha-1}(b-\lambda)^{\beta-1}$ is discussed and a closed form of the probability distribution is derived. In addition, when accuracy is desired a numerical analysis of the probability distribution is also presented. The development of the results is continued in Appendix A, as a preparation in computerizing the calculation.

Finally, an application to real data is discussed for the purpose of illustrating the model.

TABLE OF CONTENTS

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I. INTRODUCTION

Let X be a Poisson variate with the probability distribution depending on a real parameter Λ, i.e.

$$(1.1) \qquad \Pr\{X=n|\Lambda=\lambda\} = \frac{\lambda^n e^{-\lambda}}{n!}; \quad n = 0,1,2,\ldots$$

where Λ itself is a positive random variable assumed to have a prior distribution specified by the probability density $g(\lambda)$ over a range of definition R.

$$g(\lambda) \begin{cases} \geq 0 & \forall \ \lambda \in R \\ = 0 & \text{otherwise.} \end{cases}$$

$$\int_R g(\lambda)\, d\lambda = 1.$$

Depending on the functional form of $g(\lambda)$, the unconditional probability distribution of X may result in a wide variety of forms.

With regards to the application the use of (1.1) in the field of accident statistics, and acceptance sampling plans for manufactured articles based on the counting of defects, etc.... is well known. The variations in the accident liability from individual to individual in the former case; and in the latter an inevitable and continuous change in manufacturing conditions, lead to fluctuations in the Poisson parameter involved.

However, the true behavior of this random variable is not known. Thus, the distribution of Λ used so far are nothing more than approximations.

In practice, then, it is reasonable to estimate the first three or four moments of the distribution to get some general idea.

Referring to accident statistics, this method was explained and employed by Newbold in 1927 [1]. The results by her data stated that the distribution of Λ is

$$(1.2) \qquad g(\lambda) = \frac{(k/m)^k}{\Gamma(k)} \lambda^{k-1} e^{-(k/m)\lambda} \quad ; \quad \lambda \varepsilon [0;\infty)$$

where k, m > 0.

That is a Pearson type III distribution as originally suggested by Greenwood in 1919 [1], which is also known as the Gamma distribution with parameters k and k/m.

It follows that the unconditional probability of X equal to n is:

$$\Pr\{X=n\} = \int_0^\infty \Pr\{X=n|\Lambda=\lambda\} g(\lambda)\, d\lambda$$

$$(1.3) \qquad = \frac{(k/m)^k}{n!\Gamma(k)} \int_0^\infty \lambda^{n+k-1} e^{-(k/m+1)\lambda}\, d\lambda$$

$$= \frac{\Gamma(n+k)}{n!\Gamma(k)} \left(\frac{k}{m+k}\right)^k \left(\frac{m}{m+k}\right)^n$$

The random variable X is then Negative binomial distributed with parameters k and $m/(m + k)$.

Using the method of moments, to estimate these parameters, and let $\hat{k}$ and $\hat{m}$ are the estimates of k and m; $\bar{x}$ is the estimate of the mean, s^2 is the estimate of the variance of X, we have

$$\hat{m} = \bar{x}$$

$$\hat{k} = \bar{x}^2/(s^2 - \bar{x}) \quad .$$

In regards of modelling a true situation, the above model with X given Λ Poisson distributed and prior distribution Gamma $(k,k/m)$ is known as the Negative Binomial Model in accident statistics.

Employing this model, many workers in the accident statistics field achieve a good fit to real data. As an example, let us consider the following data, referred to accidents met at work by 122 shunters during a period of 6 years [1].

X	Observed Frequency
0	40
1	39
2	26
3	8
4	6
5	2
6	1

We now assume X is Poisson distributed with parameter Λ and Λ is distributed Gamma $(k,k/m)$, then apply the Negative Binomial model to the data.

Using the method of moments, with $\overline{x} = 1.27$ and $s^2 = 1.64$, to estimate k and m, we have

$$\hat{m} = 1.27$$

$$\hat{k} = 4.36$$

hence $\hat{k}/\hat{m} = 3.43$, Λ is then Gamma (4.36, 3.43) distributed.

Computing the Pr $\{X=n\}$ by (1.3), and the expected frequencies for n = 0,1,...,6 results in

X	Pr{X=n}	Expected Frequency
0	.3278	39.99
1	.3227	39.37
2	.1952	23.81
3	.0933	11.38
4	.0387	4.72
5	.0146	1.78
$\geq$ 6	.0077	.93

Performing a chi-squares goodness of fit test, results in the value of χ^2 as 1.589 with 4 degrees of freedom, hence $p \cong 80\%$.

However, the Negative Binomial model does not always give satisfactory results as in the above example. Since

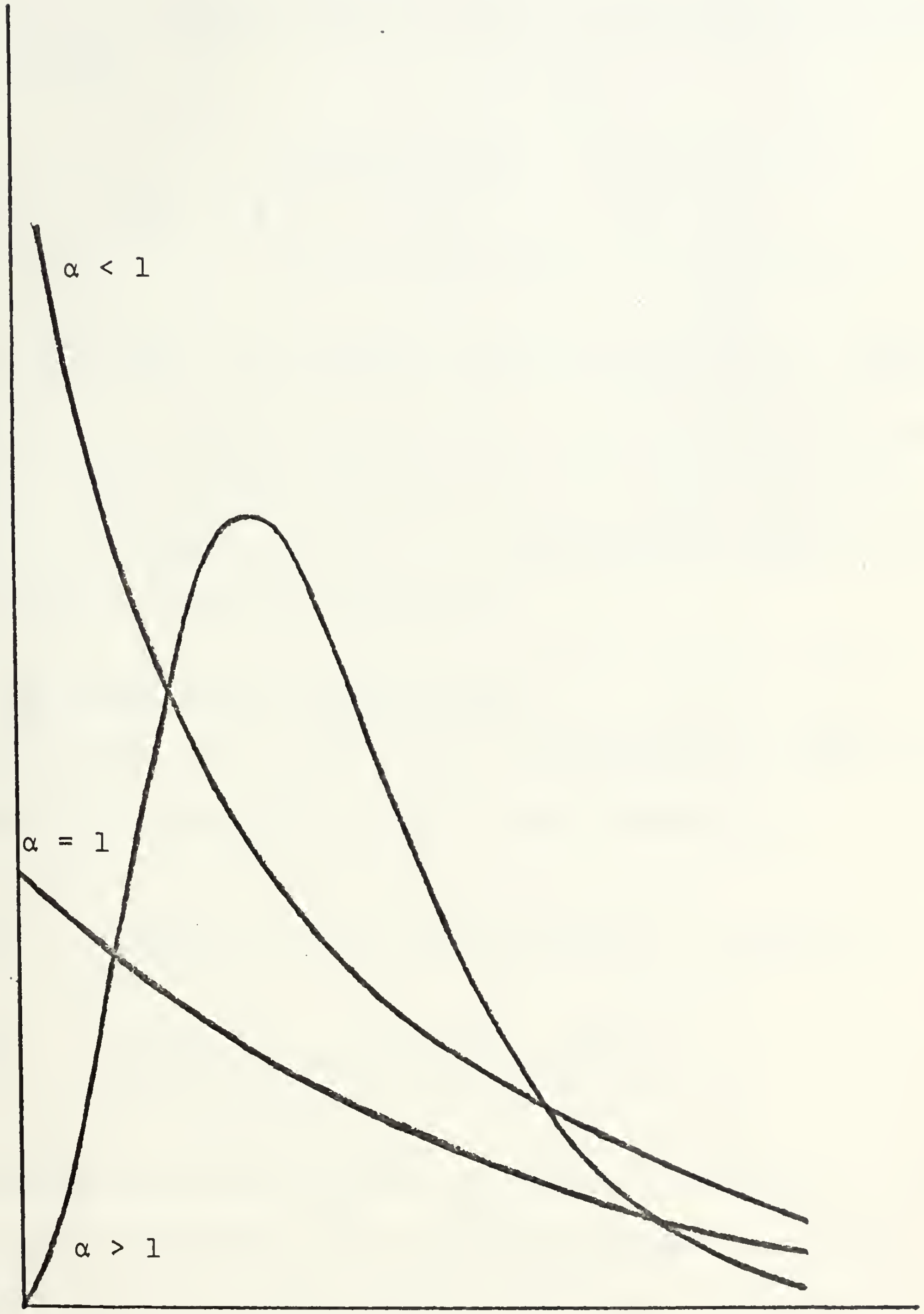
the density function specified by (1.2) is simply an approximation to true but unknown situation. In most of the cases, the approximations are based on the facts that the random variable is positive [2], and that the exponential term is included, a simplification is expected in subsequent computations. However, the latter factor also makes the assuming distribution rigid in the shape. Having two parameters, but the shape of the Gamma distribution is mostly dependent on one and can only graph three typical curves as illustrated in Table 1.

Thus, (1.2) may not reflect well the true behavior of the case where the random variable is distributed in a different manner, such as uniformly.

Working on this assumption, the purpose of this thesis is an attempt to look for a more flexible prior distribution of the parameter Λ, then study in detail the resulting unconditional distribution of the variate X.

TABLE I

Typical GAMMA curves



II. A PROPOSED PRIOR DISTRIBUTION OF λ

Define a density function $g(\lambda)$ of the random variable Λ as follows:

$$(2.1) \qquad g(\lambda) = \begin{cases} c\lambda^{\alpha-1}(b-\lambda)^{\beta-1} & ; \quad \lambda\varepsilon[0,b] \\ 0 \quad ; \quad \text{otherwise} & \end{cases}$$

where α,β and b are strictly positive, and c is a scaling factor. $g(\lambda)$ defined as in (2.1), is assumed to have the form of the Beta distribution, but the domain of definition is [0,b], instead of [0,1]. For this reason, from now on Λ is said to be Beta distributed.

A. THE SHAPE OF THE DISTRIBUTION

Let label (B) be the curve representative of $g(\lambda)$, and consider the derivative of $g(\lambda)$ with respect to λ.

$$(2.2) \qquad \frac{d}{d\lambda}\, g(\lambda) = c\lambda^{\alpha-2}(b-\lambda)^{\beta-2}\{(\alpha-1)b - (\alpha+\beta-2)\lambda\}.$$

$$(2.3) \qquad \frac{d}{d\lambda}\, g(\lambda) = 0 \quad \text{at} \quad \lambda_m = b\,\frac{\alpha-1}{\alpha+\beta-2}$$

Depending on α and β, λ_m can be within or outside of [0,b]. If $\lambda_m \varepsilon[0,b]$ then $g(\lambda)$ has a maximum (global) or minimum (global) in [0,b]. If λ_m [0,b] then $g(\lambda)$ is monotonically increasing or decreasing in [0,b].

In general, the shape of (B) near the origin is determined by the factor $\lambda^{\alpha-1}$.

Take the limit of (2.1) and (2.2), when λ approaches zero from the right. It can be seen that (B) will be tangent to the vertical axis for $\alpha < 1$; or intercept this axis for $\alpha \geq 1$. By symmetry, the factor $(b-\lambda)^{\beta-1}$ determines the shape of (B) near b.

Similarly, when λ approaches b, (B) will be tangent for $\beta < 1$; or intercept for $\beta \geq 1$ the vertical having the equation $\lambda = b$.

Thus, for different combinations of α and β, (B) may graph in a wide variety of curves. Some of the typical ones are shown in Table II, Table III, and Table IV.

B. STATISTICAL PROPERTIES OF THE DISTRIBUTION

By definition of a density function, $g(\lambda)$ must satisfy

$$\int_0^b g(\lambda)\, d\lambda = 1.$$

or in this case

$$c \int_0^b \lambda^{\alpha-1}(b-\lambda)^{\beta-1}\, d\lambda = 1.$$

Thus,

$$c = \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}$$

and

TABLE II

Typical BETA curves for $\alpha < 1$

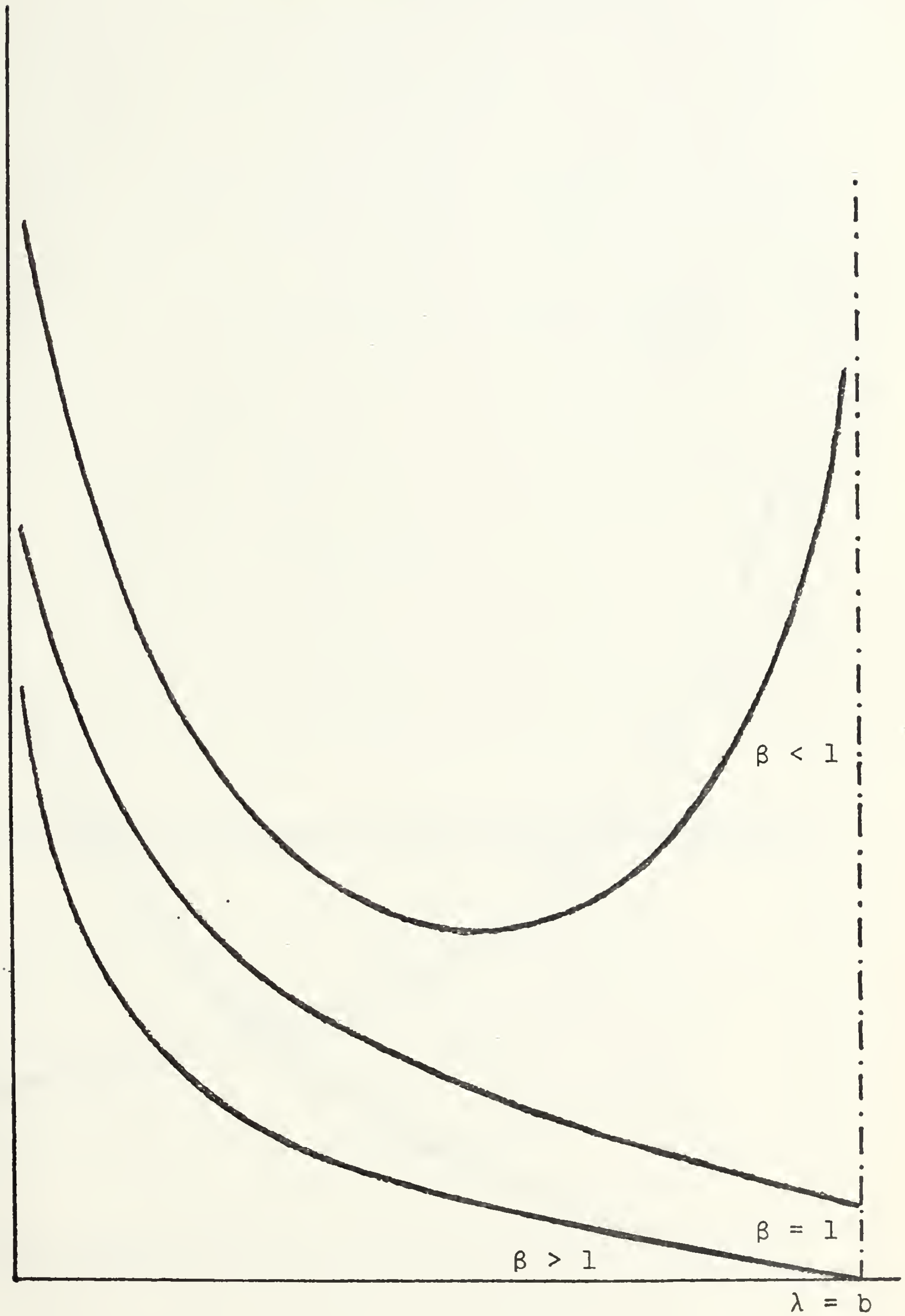


TABLE III

Typical BETA curves for $\alpha = 1$

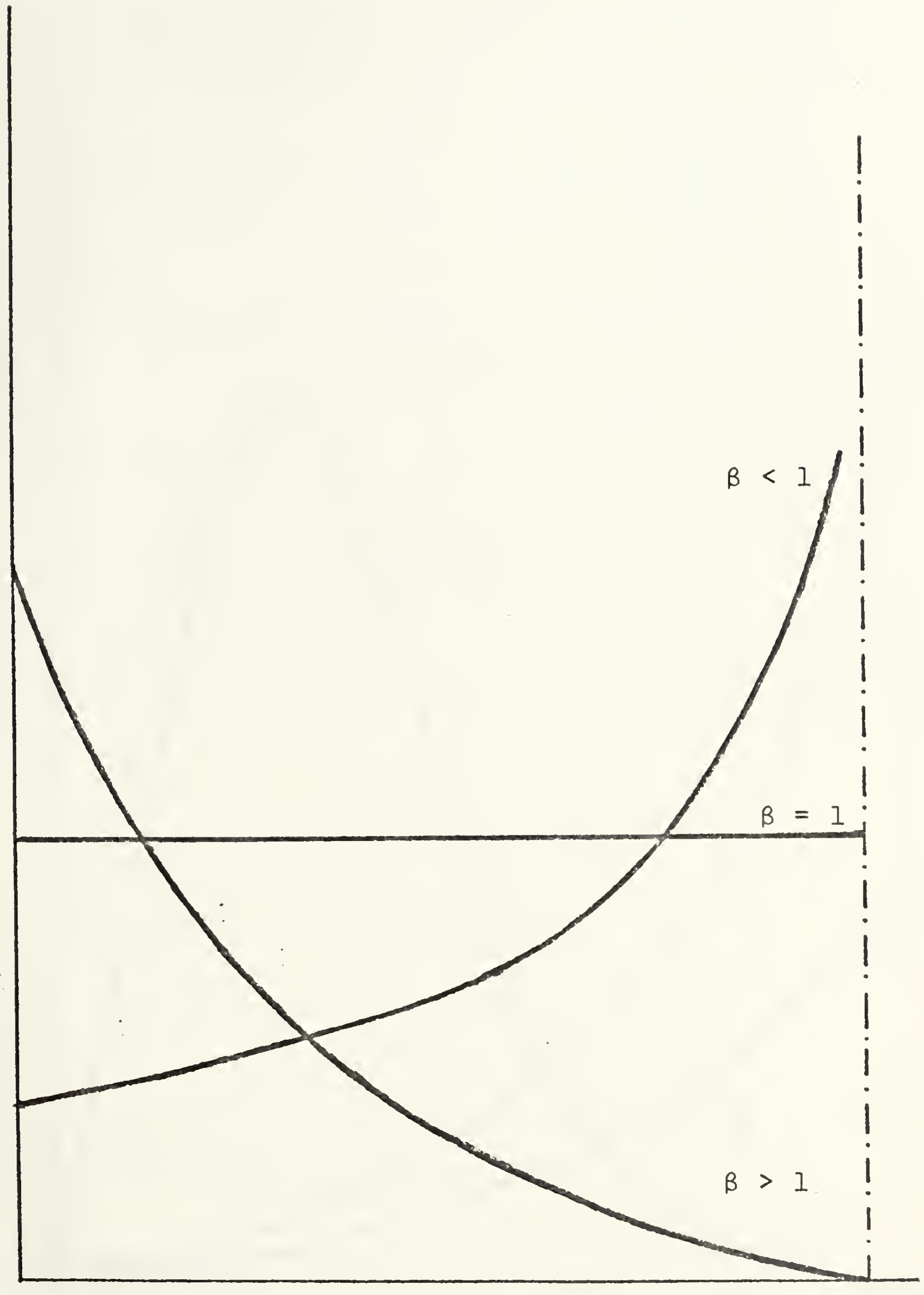
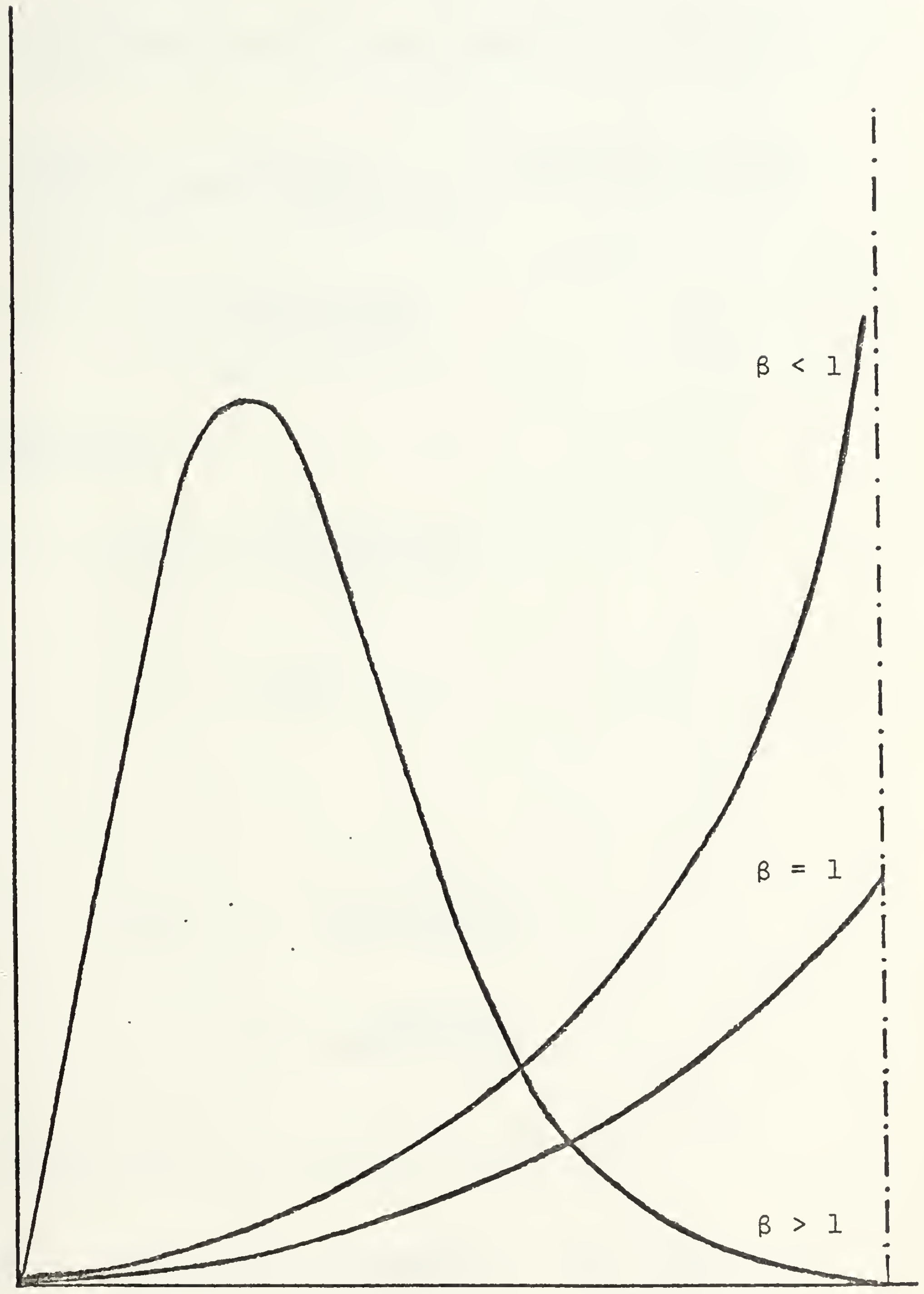


TABLE IV

Typical BETA curves for $\alpha > 1$



$$g(\lambda) = \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \lambda^{\alpha-1}(b-\lambda)^{\beta-1}$$

Let $E[\Lambda^k]$ be the kth moment about the origin of Λ,

$$E[\Lambda^k] = \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+k-1}(b-\lambda)^{\beta-1}\, d\lambda$$

$$= b^k \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(\alpha+\beta+k)}$$

In particular,

(2.4) $$E[\Lambda] = b \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}$$

$$= b \frac{\alpha}{\alpha+\beta}$$

and

(2.5) $$E[\Lambda^2] = b^2 \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)}$$

$$= b^2 \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

It follows by (2.4) and by (2.5) that

(2.6) $$V[\Lambda] = b^2 \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - b^2 \frac{\alpha^2}{(\alpha+\beta)^2}$$

On the other hand, for all $t \in (-t_0, t_0)$ the moment generating function of the random variable Λ is

$$(2.7)\quad M_\Lambda(t) = E[e^{\lambda t}] = \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha-1}(b-\lambda)^{\beta-1} e^{\lambda t}\, d\lambda.$$

which is known to exist but not particularly tractable [3].

1. The Unconditional Distribution of X

Multiplying $\Pr\{X=n|\Lambda=\lambda\}$ by $g(\lambda)$ and integrating over $[0,b]$, the unconditional probability of X equal to n is given by

$$\Pr\{X=n\} = \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \frac{\lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}}{n!} e^{-\lambda}\, d\lambda$$

$$= \frac{\Gamma(\alpha+\beta)}{n!\, b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} I(\alpha,\beta,n) .$$

By letting $t = -1$, and by substituting α by $(\alpha+n)$ in (2.7) and noting that the functional forms of $\Pr\{X=n\}$ and (2.7) are now similar. It is then immediate from (2.7) and the argument following that $\Pr\{X=n\}$ is not tractable.

However, the existence of $\Pr\{X=n\}$ can be alternatively proved as follows.

Take the sum of $\Pr\{X=n\}$ for $n = 0,1,\ldots,$ which results in

$$\sum_{n=0}^{\infty} \Pr\{X=n\} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \frac{\lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}}{n!} e^{-\lambda}\, d\lambda$$

$$= \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha-1}(b-\lambda)^{\beta-1} e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, d\lambda$$

By noting that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{\lambda}$$

so that

$$\sum_{n=0}^{\infty} \Pr\{X=n\} = \frac{\Gamma(\alpha+\beta)}{b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha-1}(b-\lambda)^{\beta-1}\, d\lambda$$

$$= \Pr\{\lambda \leq b\} = 1 \tag{2.8}$$

In addition,

$$\lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}e^{-\lambda} \geq 0 \qquad \lambda\varepsilon[0,b]$$

implying that $I(\alpha,\beta,n)$ is greater or equal to zero. Then it follows by (2.8) that $\Pr\{X=n\}$ exists as a finite positive quantity.

A closed formula of $\Pr\{X=n\}$ will be derived later in Section III.

2. Estimation Of The Parameters

Suppose observation is made on a heterogeneous population (S) a mixture of t homogeneous sub-populations during a certain period. The observed values of X are recorded, and denote X_M the maximum of these values.

Let λ_i be the value of λ of the ith sub-population, and let

$$\lambda_M = \max_i \{\lambda_i\} \quad ; \quad i = 1,2,\ldots,t.$$

and (M) be the corresponding sub-population.

Clearly, by the expression of $g(\lambda)$, b is identically equal to λ_M. However, λ_M is not known so that in practice it is estimated as follows.

Assume for a moment that the members of (M) can be identified from the heterogeneous population (S). Let m be the size of (M) and Y be the Poisson variate X, particularly referring to (M), it is immediate that

$$(2.8) \qquad E[Y] = \lambda_M$$

Now let Y_i for $i = 1,2,\ldots,m$ be the observed values of Y corresponding to the ith individual of (M). Noting that the individuals in (S), with the observed value X_M very likely belong to (M) then we can assume

$$(2.9) \qquad Y_i = X_M \quad , \quad \text{for some } i;$$

Using the method of moment to estimate λ_M, results in

$$\hat{\lambda}_M = \frac{1}{m} \sum_{i=1}^{m} Y_i$$

Further, based on the assumption of continuity made on the random variable Λ, we may repartition (S) into t+1

homogeneous sub-populations by splitting (M) into two sub-populations (M1) and (M2) in the manner so that (M2) contains a unique j^{th} member of (M) having $Y_j = X_M$. Estimating λ in (M1) and (M2), we have

$$\hat{\lambda}_{M1} = \frac{1}{m-1} \sum_{i \neq j} Y_i$$

$$\hat{\lambda}_{M2} = Y_j = X_M$$

Thus, $\hat{\lambda}_{M2} > \hat{\lambda}_{M1}$ and more general

$$\hat{\lambda}_{M2} \geq \lambda_i \qquad \text{for } i = 1,2,\ldots,M-1,M+1,\ldots,t.$$

Hence, $\hat{\lambda}_{M2}$ can be used as an estimate of λ_M, it follows that the estimate of b is

$$(2.10) \qquad \hat{b} = X_M$$

Now let $E[X]$ be the mean, and $V[X]$ be the variance of X. By elementary reasoning often used by Bayesians;

$$E[X] = E[E(X|\Lambda)]$$
$$= E[\Lambda]$$

$$V[X] = E[V(X|\Lambda)] + V[E(X|\Lambda)]$$
$$= E[\Lambda] + V[\Lambda]$$

Thus, by (2.4) and (2.5)

$$E[X] = b\,\frac{\alpha}{\alpha+\beta}$$

and

$$V[X] = b^2\,\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - b^2\,\frac{\alpha^2}{(\alpha+\beta)^2} + b\,\frac{\alpha}{\alpha+\beta}\,.$$

Denote $\bar{x}$, s^2, $\hat{\alpha}$ and $\hat{\beta}$ as the estimates of E[X], V[X], α and β respectively, then

$$\bar{x} = \hat{b}\,\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}\,,\quad \text{and}$$

(2.11) $$s^2 = \hat{b}^2\,\frac{\hat{\alpha}(\hat{\alpha}+1)}{(\hat{\alpha}+\hat{\beta})(\hat{\alpha}+\hat{\beta}+1)} - \hat{b}^2\,\frac{\hat{\alpha}^2}{(\hat{\alpha}+\hat{\beta})^2} + \hat{b}\,\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$$

Subtracting $\bar{x}$ from (2.11) and dividing by $\bar{x}^2$

(2.12) $$\frac{(\hat{\alpha}+1)(\hat{\alpha}+\hat{\beta})}{\hat{\alpha}(\hat{\alpha}+\hat{\beta}+1)} - 1 = \frac{s^2 - \bar{x}}{\bar{x}^2}$$

However,

(2.13) $$\frac{\hat{\alpha}+\hat{\beta}}{\hat{\alpha}} = \frac{\hat{b}}{\bar{x}}$$

Then, by substituting (2.13) into (2.18) and adding 1, (2.12) becomes

$$(2.14) \qquad \frac{\hat{b}(\hat{\alpha}+1)}{\bar{x}(\hat{\alpha}+\hat{\beta}+1)} = \frac{s^2 - \bar{x} + \bar{x}^2}{\bar{x}^2}$$

Now, from (2.13)

$$\hat{\alpha} + \hat{\beta} = \hat{\alpha}\,\frac{\hat{b}}{\bar{x}} ,$$

or

$$(2.15) \qquad \hat{\alpha} + \hat{\beta} + 1 = \frac{\hat{\alpha}\,\hat{b} + \bar{x}}{\bar{x}} .$$

Then, by substituting (2.15) into (2.14) to get

$$(2.16) \qquad \frac{\hat{b}(\hat{\alpha} + 1)}{\hat{b}\hat{\alpha} + \bar{x}} = \frac{s^2 - \bar{x} + \bar{x}^2}{\bar{x}^2}$$

or

$$\hat{b}(\hat{\alpha} + 1)\,\bar{x}^2 = (\hat{b}\hat{\alpha} + \bar{x})\,(s^2 - \bar{x} + \bar{x}^2) .$$

Solving for $\hat{\alpha}$ in the above equation, it comes out

$$(2.17) \qquad \hat{\alpha} = \frac{\bar{x}(\hat{b}\bar{x} + \bar{x} - \bar{x}^2 - s^2)}{\hat{b}(s^2 - \bar{x})} .$$

Substituting the value of $\hat{\alpha}$ into (2.11), to have

$$(2.18) \qquad \hat{\beta} = \hat{\alpha}\left(\frac{\hat{b}}{\bar{x}} - 1\right)$$

Now observe that estimating by by X_M, implies

(2.19) $$\hat{b} > \bar{x} \quad \text{or} \quad \frac{\hat{b}}{\bar{x}} > 1.$$

Furthermore, considering (2.9), this equation is symbolically written as

$$V[X] = V[\Lambda] + E[X]$$

Noting that the variance of a random variable is always positive, then

$$V[X] - E[X] = V[\Lambda] \geq 0, \text{ or equivalently to}$$

(2.20) $$s^2 - \bar{x} = \hat{b}^2 \frac{\hat{\alpha}(\hat{\alpha} + 1)}{(\hat{\alpha}+\hat{\beta})(\hat{\alpha}+\hat{\beta}+1)} - \hat{b}^2 \frac{\hat{\alpha}^2}{(\hat{\alpha}+\hat{\beta})^2} \geq 0$$

Also observe that by (2.17), (2.18) and (2.19), if $\hat{\alpha}$ is negative (positive) it follows that $\hat{\beta}$ is negative (positive) and vice-versa. Suppose now that $\hat{\alpha}$ and $\hat{\beta}$ both be negative. This implies that

$$\frac{\hat{\alpha} + 1}{\hat{\alpha} + \hat{\beta} + 1} < \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}$$

since

$$\frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}} < 1,$$

and leads to

$$\frac{\hat{\alpha}(\hat{\alpha} + 1)}{(\hat{\alpha} + \hat{\beta})(\hat{\alpha} + \hat{\beta} + 1)} < \frac{\hat{\alpha}^2}{(\hat{\alpha} + \hat{\beta})^2} .$$

As a result, (2.20) is negative, which is a contradiction, and thus $\hat{\alpha}$ and $\hat{\beta}$ should both be positive.

Now assume $\hat{\alpha}$ is equal to zero, $\hat{\beta}$ then is also equal to zero by (2.18). This leads to either or both of the following cases.

a. $\bar{x} = 0$

This is impossible, since the positive random variable X can not have zero mean.

b. $\bar{x} = b$

Which contradicts to (2.19)

Thus, $\hat{\alpha}$ and $\hat{\beta}$ should be both strictly positive as required in (2.1).

III. CLOSED FORMULA OF PR{X=n}

As mentioned in the last section, the exact mathematical expression of the Pr{X=n} can not be derived by the present method of integration. Approximation is then used to estimate through a closed form of the integral $I(\alpha,\beta,n)$.

First, by noting that

$$e^{-\lambda} = \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^k}{k!} .$$

Hence,

$$I(\alpha,\beta,n) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \int_0^b \lambda^{\alpha+n+k-1} (b-\lambda)^{\beta-1}\, d\lambda. \tag{3.1}$$

Now, consider the Taylor's expansion of

$$\left(1 + \frac{b}{\alpha+\beta+n}\right)^{-\alpha-n}$$

$$= 1 - (\alpha+n)\frac{b}{\alpha+\beta+n} + \sum_{k=2}^{\infty} (-1)^k \frac{b^k}{k!} \frac{(\alpha+n)\ldots(\alpha+n+k-1)}{(\alpha+\beta+n)^k} . \tag{3.2}$$

Multiplying (3.2) by $\Gamma(\beta) b^{\alpha+\beta+n-1} \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha+\beta+n)}$ to get

$$\Gamma(\beta) b^{\alpha+\beta+n-1} \sum_{k=0}^{\infty} (-1)^k \frac{b^k}{k!} \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n)(\alpha+\beta+n)^k} \tag{3.3}$$

Then, suppose for some values of α and β, the following equation is known to approximately hold

$$(\alpha+\beta+n)^k \cong (\alpha+\beta+n)\ldots(\alpha+\beta+n+k-1)$$

it follows then that

$$\Gamma(\alpha+\beta+n)(\alpha+\beta+n)^k \cong \Gamma(\alpha+\beta+n+k) \tag{3.4}$$

and thus (3.3) becomes

$$\Gamma(\beta)b^{\alpha+\beta+n-1} \sum_{k=0}^{\infty} (-1)^k \frac{b^k\Gamma(\alpha+n+k)}{k!\Gamma(\alpha+\beta+n+k)} \tag{3.5}$$

Taking into account (3.4), then comparing (3.5) with (3.1), it leads to

$$(1 + \frac{b}{\alpha+\beta+n})^{-\alpha-n} \cong [\Gamma(\beta)b^{\alpha+\beta+n-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+\beta+n)}]^{-1} I(\alpha,\beta,n)$$

recalling from the last section, $\Pr\{X=n\}$ is written as

$$\Pr\{X=n\} = \frac{\Gamma(\alpha+\beta)}{n!b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} I(\alpha,\beta,n)$$

Hence,

$$\Pr\{X=n\} \cong \frac{b^n\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} (1 + \frac{b}{\alpha+\beta+n})^{-\alpha-n} \tag{3.6}$$

It can be seen that formula (3.6) above is an overestimate of $Pr\{X=n\}$, i.e.

$$Pr\{X=n\} = \frac{b^n\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \left(1 + \frac{b}{\alpha+\beta+n}\right)^{-\alpha-n} - \varepsilon_n$$

where ε_n is a positive quantity, depending on α,β and n.

In fact, let

$$(3.7) \qquad T_1^k = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \frac{b^k\Gamma(\alpha+n+k)}{k!\Gamma(\alpha+\beta+n)(\alpha+\beta+n)^k}$$

$$(3.8) \qquad T_2^k = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \frac{b^k\Gamma(\alpha+n+k)}{k!\Gamma(\alpha+\beta+n+k)}$$

Multiplying both numerator and denominator of T_1^k by $(\alpha+\beta+n)\ldots(\alpha+\beta+n+k-1)$ it results in

$$(3.9) \qquad T_3^k = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \frac{b^k\Gamma(\alpha+\beta+n)(\alpha+\beta+n)\ldots(\alpha+\beta+n+k-1)}{k!\Gamma(\alpha+\beta+n+k)(\alpha+\beta+n)^k}$$

Taking the difference δ_k of (3.9) and (3.8), and noting that for $\delta_0 = \delta_1 = 0$ then for $k \geq 2$

$$\delta_k = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \frac{b^k\Gamma(\alpha+n+k)}{k!\Gamma(\alpha+\beta+n+k)} \left\{ \frac{(\alpha+\beta+n)\ldots(\alpha+\beta+n+k-1)}{(\alpha+\beta+n)^k} - 1 \right\}$$

$$= \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \frac{b^k\Gamma(\alpha+n+k)}{k!\Gamma(\alpha+\beta+n+k)} \left\{ \prod_{j=0}^{k-1} \left(1 + \frac{j}{\alpha+\beta+n}\right) - 1 \right\}$$

Thus, for $k \geq 2$, δ_k increases as long as the numerator is larger than the denominator, then for some k, it converges to zero. The speed of the convergence is proportional to $\alpha + \beta$.

It follows that

$$\varepsilon_n = \sum_{k=2}^{\infty} (-1)^k \delta_k$$

Hence, for $\alpha + \beta$ large enough, ε_n is small, (3.6) is then a good approximation of $\Pr\{X=n\}$.

In practice, (3.6) can be applied vastly together with some correction. Before doing so, however δ_k should be expressed more explicitly.

Consider

$$\prod_{j=0}^{k-1} \left(1 + \frac{j}{\alpha+\beta+n}\right) - 1,$$

and assume that $\frac{1}{\alpha+\beta+n}$ is small enough so that

$$\prod_{j=0}^{k-1} \left(1 + \frac{j}{\alpha+\beta+n}\right) - 1 \cong \frac{1}{\alpha+\beta+n} + \frac{2}{\alpha+\beta+n} + \ldots + \frac{k-1}{\alpha+\beta+n}$$

$$\cong \frac{(k-1)k}{2(\alpha+\beta+n)}$$

hence,

$$\delta_k = \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+n)}{2(\alpha+\beta+n)n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \left(\frac{b^k(\alpha+n)\ldots(\alpha+n+k-1)}{k!(\alpha+\beta+n)\ldots(\alpha+\beta+n+k-1)}\right)(k-1)k$$

and

$$\varepsilon_n = \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+n)}{2(\alpha+\beta+n)n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \left\{ \sum_{k=2}^{\infty} (-1)^k \frac{b^k(\alpha+n)\ldots(\alpha+n+k-1)}{k!(\alpha+\beta+n)\ldots(\alpha+\beta+n+k-1)} (k-1)k \right\}$$

The factor in brackets does not depend greatly on n. It approximately has the common value σ for some small n. Thus,

$$\varepsilon_n = \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+n)}{2(\alpha+\beta+n)n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \sigma ,$$

now suppose there is an N, such that

a. $S = \sum_{n=0}^{N} \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+n)}{n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \left(1 + \frac{b}{\alpha+\beta+n}\right)^{-\alpha-n} > 1$

b. $\Pr\{X=n\}$ is close to zero for all $n > N$

Then,

$$S - 1 = \sum_{n=0}^{N} \varepsilon_n$$

$$= \sigma \sum_{n=0}^{N} \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+n)}{2(\alpha+\beta+n)n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} .$$

Solving for σ in this equation, it results in

$$\sigma = (S-1) \left[\sum_{n=0}^{N} \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+n)}{2(\alpha+\beta+n)n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \right]^{-1}$$

Substituting σ back into (3.10), to get the approximative ε_n, and finally, with the correction, (3.6) is modified as

$$(3.11) \qquad \Pr\{X=n\} \cong \frac{b^n \Gamma(\alpha+\beta)\Gamma(\alpha+\beta n)}{n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)} \left\{ \left(1 + \frac{b}{\alpha+\beta+n}\right)^{-\alpha-n} - \frac{\sigma}{2(\alpha+\beta+n)} \right\}$$

The application of the above results will be seen in Section V. As is shown, the estimate of $\Pr\{X=n\}$, using (3.11) agrees to two decimal digits, in overall, with the results using (1.3), in Section I, for $b = 6$ and $\alpha + \beta = 13.009782$.

IV. NUMERICAL TECHNIQUE IN APPROXIMATING PR X=n

In the last section, a closed formula of Pr{X=n} was derived. Although it proved itself a fairly good estimation, the error involved was uncontrollable, and thus the validity of the proposed distribution of Λ in Section II cannot be judged. Hence, numerical techniques of integration are suggested to evaluate the Pr{X=n} through the integral $I(\alpha,\beta,n)$. Several methods have been considered, such as Gaussian quadrutures and Romberg's extrapolation to the limit [4]. However, the direct application of these methods leads to unsatisfactory programming considerations and unbounded error estimates.

Remember that Pr{X=n} can be written as

$$\Pr\{X=n\} = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \sum_{k=0}^{\infty} (-1)^k \frac{b^k\Gamma(\alpha+n+k)}{k!\Gamma(\alpha+\beta+n+k)}$$

equivalent to a series of alternate sign terms which converges monotonically to zero for large enough k. This behavior guarantees that Pr{X=n} can be estimated by summing the first K terms, and the absolute error is no larger than the absolute value of the K + 1st term, i.e.

$$\frac{b^n\Gamma(\alpha+\beta)b^{K+1}\Gamma(\alpha+n+K+1)}{n!\Gamma(\alpha)(K+1)!\Gamma(\alpha+\beta+n+K+1)} .$$

It seemed to be a powerful method, however the convergence is rather slow. Moreover, if the estimation is done by means of a computer, the precision will be altered by the propagation of the error in repeated subtraction of nearly equal terms in the alternating series.

Thus, a special technique is required which is expected to improve the convergence and also eliminate the error propagation.

This technique is simply a variant of the Gaussian quadruture method.

A. CONVERGENCE ACCELERATION

The idea is to approximate the factor $e^{-\lambda}$ in the integral

$$\Pr\{X=n\} = \frac{\Gamma(\alpha+\beta)}{n!\,b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}e^{-\lambda}d\lambda$$

with an orthogonal polynomial $p_s(\lambda)$ of degree s by least-squares approximation method [4]. In other words, it is to find $p_s(\lambda)$ so that

(4.1) $$E(s) = \int_0^b \{e^{-\lambda} - p_s(\lambda)\}^2\, d\lambda \text{ is minimized.}$$

From the theory of this method $p_s(\lambda)$ can be expressed as

(4.2) $$p_s(\lambda) = \sum_{j=0}^{s} d_j P_j(\lambda)$$

where the $\{P_j\}$ is a sequence of ortho-polynomials of degree j and satisfies the properties below:

$$P_0(\lambda) = 1.$$

(4.3) $$\int_0^b P_k(\lambda)\, P_j(\lambda)\, d\lambda = \begin{cases} S_j & ; \quad \text{if } k = j. \\ 0 & ; \quad \text{if } k \neq j. \end{cases}$$

(4.4) $$d_j = \frac{1}{S_j} \int_0^b e^{-\lambda}\, P_j(\lambda)\, d\lambda$$

This method also provides a recurrence formula of $P_{j+1}(\lambda)$ in terms of $P_j(\lambda)$ and $P_{j-1}(\lambda)$ which is

(4.5) $$P_{j+1}(\lambda) = (\lambda - B_j)\, P_j(\lambda) - C_j P_{j-1}(\lambda) \quad ;$$

where B_j and C_j are defined as

(4.6) $$B_j = \frac{1}{S_j} \int_0^b \lambda P_j(\lambda)^2\, d\lambda$$

and

(4.7) $$C_j = \begin{cases} \dfrac{S_j}{S_{j-1}} & \text{for} \quad j \geq 1 \\ 0 & \text{for} \quad j = 0 \end{cases}$$

For the sake of simplicity, let now denote $P_j(\lambda)$ by P_j and $p_s(\lambda)$ by simply p_s. Then, taking into account (4.1) and (4.2), E(s) was expressed as

$$E(s) = \int_0^b \{e^{-\lambda} - p_s\}^2 \, d\lambda$$

$$= \int_0^b e^{-2\lambda} d\lambda + \sum_{j=0}^{s} \sum_{k=0}^{s} d_j d_k \int_0^b p_j p_k d\lambda - 2 \sum_{j=0}^{s} d_j \int_0^b e^{-\lambda} P_j d\lambda .$$

by (4.3) and (4.4), E(s) is reduced to

$$E(s) = 1/2 \, (1 - e^{-2b}) - \sum_{j=0}^{s} d_j^2 \, S_j$$

Thus, for a value of b, E(s) is decreasing when p_s takes a higher degree, and as b increases s should be also increased in order to keep E(s) as small as desired.

As an example, some of the values of E(s) were displayed in Table V.

TABLE V

The least-squares error E(s), for b = 6

E(0)	=	.33415549	
E(1)	=	.10972447	
E(2)	=	.20317197	10^{-1}
E(3)	=	.23624092	10^{-2}
E(4)	=	.18656389	10^{-3}
E(5)	=	.10601508	10^{-4}
E(6)	=	.45284147	10^{-6}
E(7)	=	.15041719	10^{-7}
E(8)	=	.39915586	10^{-9}
E(9)	=	.86497233	10^{-11}
E(10)	=	.15588653	10^{-12}

B. GENERATION OF THE SEQUENCE $\{P_j\}$

By definition, P_j can be written as

$$P_j = \sum_{k=0}^{j} A_k^{\,j} \lambda^k \quad ; \tag{4.8}$$

where the $A_k^{\,j}$'s are the constant coefficients of the λ^k's. It is now to generate explicitly the $\{P_j\}$, i.e., to find the $A_k^{\,j}$'s in terms of b. Before doing so, let us express S_j, B_j, and d_j in a computable form.

Taking into account (4.8), (4.3) is then written as

$$\begin{aligned} S_j &= \sum_{k=0}^{j} \sum_{\ell=0}^{j} \int_0^b A_k^{\,j} A_\ell^{\,j} \lambda^{k+\ell}\, d\lambda \\ &= \sum_{k=0}^{j} \sum_{\ell=0}^{j} \left. \frac{A_k^{\,j} A_\ell^{\,j} \lambda^{k+\ell+1}}{k+\ell+1} \right|_0^b \\ &= \sum_{k=0}^{j} \sum_{\ell=0}^{j} \frac{A_k^{\,j} A_\ell^{\,j} b^{k+\ell+1}}{k+\ell+1} \end{aligned} \tag{4.9}$$

Similarly, (4.6) becomes

$$\begin{aligned} B_j &= \frac{1}{S_j} \sum_{k=0}^{j} \sum_{\ell=0}^{j} \int_0^b A_k^{\,j} A_\ell^{\,j} \lambda^{k+\ell+1}\, d\lambda \\ &= \frac{1}{S_j} \sum_{k=0}^{j} \sum_{\ell=0}^{j} \left. \frac{A_k^{\,j} A_\ell^{\,j} \lambda^{k+\ell+2}}{k+\ell+2} \right|_0^b \\ &= \frac{1}{S_j} \sum_{k=0}^{j} \sum_{\ell=0}^{j} \frac{A_k^{\,j} A_\ell^{\,j} b^{k+\ell+2}}{k+\ell+2} \quad . \end{aligned} \tag{4.10}$$

and finally (4.4) becomes

$$(4.11) \qquad d_j = \frac{1}{S_j} \sum_{k=0}^{j} A_k^{\,j} \int_0^b \lambda^k e^{-\lambda}\, d\lambda$$

$$= \frac{1}{S_j} \sum_{k=0}^{j} A_k^{\,j} \{- \sum_{r=0}^{k} e^{-b} \frac{k!}{(k-r)!} b^{k-r} + k!\}$$

In particular, for j = 0 applying the results just developed to get

$$S_0 = b \; ; \quad B_0 = b/2 \; ; \quad \text{and} \quad d_0 = (1 - e^{-b})/b$$

Then, by (4.5)

$$P_1 = (\lambda - b/2) \; .$$

With the expression of P_1 as above, applying again (4.9), (4.10) and (4.11) for j = 1, it results in

$$S_1 = b^3/12 \; ; \quad B_1 = b/2 \quad \text{and}$$

$$d_1 = -e^{-b}(12/b^3 - 6/b^2) + 12/b^3 - 6/b^2$$

For $j \geq 1$, P_{j+1} is generated recursively by (4.5) and by means of a Fortran-Formac program [5], which has the capability of manipulating symbolic mathematical expressions. This program also computes S_{j+1}, B_{j+1} and d_{j+1} directly by

(4.9), (4.10) and by (4.11). Some of the first P_j, so generated, are displayed in Table VI.

The following is the Algorithm, which the Fortran-Formac program mentioned above is based on.

ALGORITHM 1.

Step 1. Read in P_0, S_0, C_0, d_0

Read in P_1, S_1, C_1, d_1

Set $j = 1$

Step 2. Compute P_{j+1} by (4.5)

Compute S_{j+1} by (4.9)

Compute B_{j+1} by (4.10)

Compute d_{j+1} by (4.11)

Step 3. Updating j by 1

Go to Step 2.

With Fortran-Formac programming, given P_{j-1}, P_j, we can obtain P_{j+1} at any desired j. However, in practice, if b is known based on (4.5), the coefficients A_k^j's can be obtained with Fortran programming alone as follows.

$$A_0^{j+1} = -B_j A_0^j - C_j A_0^{j-1} \quad ;$$

for $1 \leq m \leq j-2$

$$A_m^{j+1} = -B_j A_m^j - C_j A_m^{j-1} + A_{m-1}^j \; ;$$

and

$$A_{j-1}^{j+1} = - B_j A_{j-1}^j + A_{j-2}^j \quad ;$$

$$A_j^{j+1} = A_{j-1}^j \; .$$

TABLE VI

P_j and d_j for $j = 2,3$.

$$P_2 = \lambda^2 \quad -b\lambda \quad +b^2/6.$$

$$P_3 = \lambda^3 \quad -3b\lambda^2/2 + 3b^2\lambda/5 - b^3/20.$$

$$d_2 = -e^{-b}(360/b^5 + 180/b^4 + 30/b^3) + 360/b^5$$
$$- 180/b^4 + 30/b^3.$$

$$d_3 = -e^{-b}(16800/b^7 + 8400/b^6 + 1680/b^5 + 140/b^4)$$
$$+ 16800/b^7 - 8400/b^6 + 1680/b^5 - 140/b^4.$$

But, the A_k^j's computed with Fortran programming were not as precise as those derived by Fortran-Formac, since it was dealing with a large amount of computation, while the A_k^j's are evaluated directly from symbolic expressions in Fortran Formac programming. However, the differences were negligable when j was less than 10.

Thus, taking account of (4.8) and with the A_k^j's found as above, p_s is now written as

$$(4.12) \qquad p_s = \sum_{j=0}^{s} d_j \sum_{k=0}^{j} A_k^j \lambda^k$$

Now, let P(n,s) be the estimate of Pr{X=n}, approximate $e^{-\lambda}$ with p_s. Taking account of (4.12),

$$P(n,s) = \frac{\Gamma(\alpha+\beta)}{n!b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{s} d_j \sum_{k=0}^{j} A_k^j \int_0^b \lambda^{\alpha+n+k-1}(b-\lambda)^{\beta-1} d\lambda$$

Further, let $t_k^j = d_j A_k^j$. Then,

$$P(n,s) = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \sum_{j=0}^{s} \sum_{k=0}^{j} t_k^j b^k \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n+k)} .$$

and P(n,s) above was computed by means of an iterative method, based on the following algorithm.

ALGORITHM 2.

Step 1. Set j = 0

Let $P(n,s) = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} t_0^o \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n+k)}$

Step 2. Set $j = j + 1$

Compute $T(j) = \frac{b^n \Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \sum_{k=0}^{j} t_k^j \, b^k \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n+k)}$.

Set $P(n,s) = P(n,s) + T(j)$

Step 3. If $|T(j)| > r$ Go to Step 2

If $|T(j)| \leq r$ Stop

It is noted here, that Step 2 was called the $J + 1^{th}$ iteration. The stopping criteria in Step 3 is only for the purpose of fixing the idea, it was replaced later by a more appropriate one discussed in the next subsection, when the error in $P(n,s)$ is discussed.

C. ERROR ESTIMATION

Let $\varepsilon(n,s)$ be the error committed in approximating $P(n,s)$ to $\Pr\{X=n\}$. Thus,

$$(4.13) \quad \varepsilon(n,s) = \Pr\{X=n\} - P(n,s)$$

$$= \frac{\Gamma(\alpha+\beta)}{n!b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}(e^{-\lambda}-p_s)d\lambda$$

Now by the mean value theorem of integration, and by noting that $\lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}$ does not change sign in $[0,b]$, then there exists a $\lambda^* \varepsilon [0,b]$, which depends on $\lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}$ and on $e^{-\lambda}-p_s(\lambda)$, so that

$$(4.14)\quad \varepsilon(n,s) = \{e^{-\lambda^*} - p_s(\lambda^*)\ \frac{\Gamma(\alpha+\beta)}{n!\,b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}d\lambda$$

In general, such a λ^* in (4.14) is not known, hence $\varepsilon(n,s)$ cannot be found. However, by using the Cauchy-Schwarz inequality the bound $u(n,s)$ of the error is derived as follows.

$$|\varepsilon(n,s)| \leq u(n,s) = \frac{\Gamma(\alpha+\beta)}{n!\,b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \left[\int_0^b \lambda^{2(\alpha+n-1)}(b-\lambda)^{2(\beta-1)}\right]^{\frac{1}{2}}$$

$$\left[\int_0^b (e^{-\lambda}-p_s)^2\ d\lambda\right]^{\frac{1}{2}}$$

Recalling that,

$$E(s) = \int_0^b (e^{-\lambda}-p_s)^2\ d\lambda = \frac{1}{2}\,(1-e^{-2b}) - \sum_{j=0}^{s} d_j^2\ S_j.$$

and that

$$\int_0^b \lambda^{2(\alpha+n-1)}(b-\lambda)^{2(\beta-1)}d\lambda = b^{2(\alpha+\beta+n-1)-1}\ \frac{\Gamma(2\alpha+2n-1)\Gamma(2\beta-1)}{\Gamma(2\alpha+\ 2\beta+2n-2)}$$

then

$$(4.15)\quad u(n,s) = \frac{b^{n-\frac{1}{2}}\Gamma(\alpha+\beta)}{n!\,\Gamma(\alpha)\Gamma(\beta)}\left[\frac{\Gamma(2\alpha+2n-1)\Gamma(2\beta-1)}{\Gamma(2\alpha+2\beta+2n-1)}\ E(s)\right]^{\frac{1}{2}}$$

On the other hand, the error $\varepsilon(n,s)$ in (4.13) can be alternatively written as:

$$\varepsilon(n,s) = \frac{\Gamma(\alpha+\beta)}{n!b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}e^{-\lambda}\, d\lambda$$

$$- \frac{\Gamma(\alpha+\beta)}{n!b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}p_s\, d\lambda.$$

Now, applying the mean value theorem of integration to the two above integrals, and let

$$V = \frac{\Gamma(\alpha+\beta)}{n!b^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \int_0^b \lambda^{\alpha+n-1}(b-\lambda)^{\beta-1}\, d\lambda$$

$$= \frac{b^n\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{n!\Gamma(\alpha)\Gamma(\alpha+\beta+n)}, \quad \text{then}$$

$$(4.16) \qquad \varepsilon(n,s) = \{e^{-\lambda_0} - p_s(\lambda_1)\}\, V$$

where λ_0 and λ_1 $\varepsilon[0,b]$.

Adding and subtracting to (from) (4.16) the quantity $e^{-\lambda_1} V$, we have,

$$(4.17) \qquad \varepsilon(n,s) = \{e^{-\lambda_1} - p_s(\lambda_1)\}\, V + \{e^{-\lambda_0} - e^{-\lambda_1}\}\, V$$

$$= e^{-\lambda_1} V - P(n,s) + \{e^{-\lambda_0} - e^{-\lambda_1}\}\, V$$

Further, assume that $\lambda_0 \simeq \lambda_1$, then (4.17) becomes

$$(4.18) \qquad \varepsilon(n,s) \cong e^{-\lambda_1} V - P(n,s)\ .$$

It can be seen that (4.18) is a good estimate for $\varepsilon(n,s)$ only when the above assumption holds.

The λ_1 in (4.18) can be found by solving

$$(4.19) \qquad f(\lambda) = p_s(\lambda) - \frac{P(n,s)}{V} = 0$$

Based on the above analysis, algorithm 2 is now modified as,

Step 1. Set $j = 0$

Set $P(n,s) = \frac{b^n \Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \; t_0^o \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n+k)}$

Step 2. Set $j = j + 1$

Compute $T(j) = \frac{b^n \Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \sum_{k=0}^{j} t_k^j b^k \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n+k)}$

$P(n,s) = P(n,s) + T(j)$

Step 3. Compute $u(n,s)$

Step 4. If $u(n,s) \leq r$ Stop

If $u(n,s) > r$ Go to Step 2

However, for the purpose of comparison, the approximate $\varepsilon(n,s)$ as in (4.18) is also computed, and it is suggested that (4.19) be solved by Newton iterative method which is suitable for this problem [4]. In fact, $f(\lambda)$ is continuously differentiable in [0,b] as the Newton method required. In addition, the derivative of $f(\lambda)$ is easily computable.

Taking into account (4.12), then

$$f(\lambda) = \sum_{j=0}^{s} \sum_{k=0}^{j} t_k^j \lambda^k - \frac{P(n,s)}{V}$$

it follows that,

$$\frac{d}{d\lambda} f(\lambda) = \sum_{j=1}^{s} \sum_{k=1}^{j} k\, t_k^j \lambda^{k-1} \tag{4.20}$$

Moreover, (4.20) evaluated at λ_1 does not vanish as $p_s(\lambda)$ approaches $e^{-\lambda}$.

On the other hand, a starting point $\lambda^{(o)}$ for the Newton iterations is nicely provided by:

$$e^{-\lambda^{(o)}} = \frac{P(n,s)}{V} \tag{4.21}$$

In such conditions, the Newton methods are very efficient, i.e., converge rather fast (quadratically) to λ_1.

For a detailed treatment of the analysis, the reader is referred to Appendix A.

V. AN ILLUSTRATION

Consider the following example.

A sample of 621 white male children is observed, when they were from 8 to 11 years old. The data reported here is a part of a large study of childhood accidents conducted by the State of California Department of Public Health. Subjects to study were those children whose families subscribed to the Kaiser Foundation Health Plan [6].

The number of injuries and the observed frequency were tabulated below.

X	Observed Frequency
0	240
1	192
2	107
3	52
4	17
5	9
6 and more	4

For purposes of comparison, let us first apply the Negative Binomial to these data, i.e. assuming that X is a Poisson variate with parameter Λ, and Λ itself is Gamma distributed.

A. THE NEGATIVE BINOMIAL MODEL

1. Distribution of Λ

Using the standard method to estimate the mean and the variance of X results in

$$\bar{x} = 1.13$$

and

$$s^2 = 1.52$$

Hence, by (1.4) and (1.5),

$$m = 1.13$$

and

$$k = 3.24, \text{ thus } k/m = 2.87$$

It follows that from (1.2), $g(\lambda)$ is Gamma (3.24,2.87).

The mean and the mode are respectively,

$$E[\Lambda] = 1.13$$

$$\lambda_m = .785$$

The curve representative (C) of $g(\lambda)$ was shown in Table VII.

2. <u>The Unconditional Probability of X = n</u>

With the values of k and as above, and by (1.3), $Pr\{X=n\}$ is

$$\frac{\Gamma(3.24 + n)}{n!} .15 \times .26^n$$

Then, for n = 0,1,2,3,4,5,6 and more:

n	Pr{X=n}	Expected Frequency
0	.38063	236.37
1	.31375	197.38
2	.17374	107.88
3	.07836	48.59
4	.03149	19.55
5	.01176	7.30
≥ 6	.00627	3.89

$\chi^2 = 1.179$ with 4 degrees of freedom

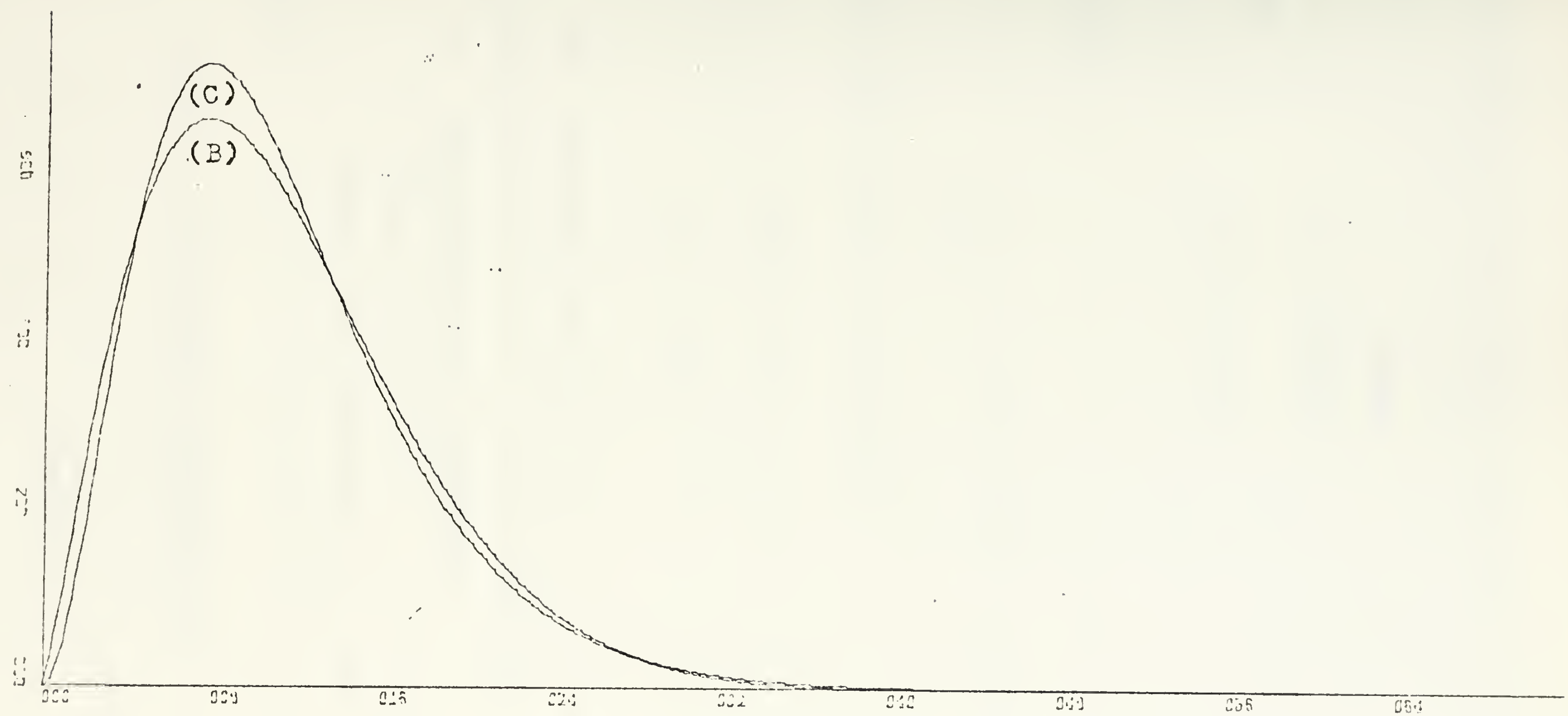
$75\% < p < 90\%$

B. APPLICATION OF SECTIONS II, III, IV.

1. <u>Distribution of Λ</u>

Assume now $g(\lambda)$ is of the form (2.1), and assume further that the probability of X is greater than 6 is small, then by (2.9)

$$b = 6$$



X-SCALE=8.00E-01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
TABLE VII
(B) VERSUS (C)

It follows by (2.17) and by (2.18) that

$$\alpha = 2.440642$$

$$\beta = 10.569140$$

and

$$\alpha + \beta = 13.009782$$

hence,

$$g(\lambda) = .13\ 10^{-6}\ \lambda^{1.440642}\ (6-\lambda)^{9.56914}$$

with mean and mode respectively,

$$E[\Lambda] = 1.13$$

$$\lambda_m = .785$$

The curve representative (B) is shown in Table VII. It is noted that (B) and (C) have the same mode, $\lambda_m = .785$.

2. The Unconditional Probability

a. Using Closed Formula.

Substituting the values of α and β found above in (3.6)

$$\Pr\{X=n\} = \frac{6^n\ 3.85\ 10^8\ \Gamma(2.440642 + n)}{n!\ \Gamma(13.009782 + n)}$$

$$\left(1 + \frac{6}{13.009782 + n}\right)^{-2.440642-n}$$

Then, for n = 0, 1, 2, 3, 4, 5 and 6, the rough estimates of the probabilities of the unconditional of X equal to n are the following

n	Pr{X=n}
0	.39629
1	.33017
2	.18628
3	.18628
4	.08685
5	.01338
6	.00463

Now applying the correction to the above results. By (3.11) and (3.12),

$$S = 1.0533$$
$$S - 1 = .0533$$

and

$$\alpha = .4001$$

Thus, the results should be corrected as

n	Corrected
0	.38091
1	.31049
2	.17522
3	.08071
4	.03279
5	.01211
6	.00414

It can be seen that with the correction, i.e., using formula (3.13), the estimates of the probabilities of X equal to n, can be compared within two decimal digits to the results found by the Negative Binomial model. However, the significance of these figures is uncontrollable, hence there is no use to perform a goodness of fit test here.

b. Using Numerical Technique

Fixing in advance $r = .156\ 10^{-5}$, then after 11 iterations P(n,s) converge to the following figures.

n	P(n,s)	u(n,s)	Expected Frequency
0	.383299	$2.0\ 10^{-7}$	238.03
1	.313539	$2.9\ 10^{-7}$	194.71
2	.173662	$2.1\ 10^{-7}$	107.84
3	.079539	$1.2\ 10^{-7}$	49.39
4	.032189	$6.2\ 10^{-8}$	19.39
5	.011869	$2.8\ 10^{-8}$	7.37
6	.004057	$1.1\ 10^{-8}$	2.51

Based on the above results, and by noting that the sum of P(n,s) from n = 0,1,... equals to one, a chi-square goodness of fit test is performed, the results are $\chi^2 = 1.035$ with 3 degrees of freedom and $75\% < p < 90\%$.

It is noted that the values of the P(n,s) and u(n,s) are rounded off to the significant digits. For the convergence and the actual values of P(n,s), u(n,s) and $\varepsilon(n,s)$ the reader is referred to Appendix B.

The summary of the results obtained in this section are shown in Table VII.

TABLE VIII

SUMMARY OF RESULTS

X	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
0	.38063	.38091	.3832990	240	236.37	236.54	238.03
1	.31785	.31409	.3135385	192	197.38	195.04	194.71
2	.17374	.17522	.1736619	107	107.88	108.81	107.84
3	.07826	.08071	.0795389	52	48.59	50.12	49.39
4	.03149	.03279	.0321885	17	19.55	20.36	19.98
5	.01176	.01211	.0118698	9	7.30	7.52	7.37
6	.00627	.00417	.0059066	4	3.89	2.61	3.66

Note:

(i) : $Pr\{X=n\}$ (negative binomial)

(ii) : $Pr\{X=n\}$ (closed formula 2 with correction)

(iii) : $Pr\{X=n\}$ (numerical technique)

(iv) : Observed frequency.

(v) : Expected frequency (negative binomial)

(vi) : Expected frequency (closed formula 2 with correction)

(vii) : Expected frequency (numerical technique)

VI. CONCLUSIONS

As is shown in Section V, an application to the real situation of the new model is presented. In it we assume that the data is Poisson distributed with parameter Λ and the parameter itself is also a random variable with density function of the form as specified in (2.1). The validity of the model is then made by performing a chi-squares goodness of fit test. As the results, $\chi^2 = 1.035$ with 3 degrees of freedom and $75\% < p < 90\%$.

On the other hand, modelling the set of data as Negative binomial model, and using the same validation procedure the results are $\chi^2 = 1.179$ with 4 degrees of freedom and again $75\% < p < 90\%$.

Comparing the two results, and referring particularly to the expected frequencies, it can be seen that those of the new model are closer to the real data. However, because the distribution specified in (2.1) has 3 parameters, i.e., 1 parameter more than the Gamma, causing 1 degree of freedom less in the chi-squares goodness of fit test the results do not differ significantly. Availability of an extra parameter is one of the major advantages of the new model in regards to the modelling of any real situation. However, using the maximum of the observed values of X in a single period as an estimate of the parameter b in Section II, is not an adequate method. It leads to difficulties in extending the

model in order to predict the distribution of X in the next period.

Moreover, the exact mathematical expression of Pr{X=n} cannot be derived, hence the knowledge about the statistical properties of X is limited to the mean and the variance.

In Section III, a closed formula of Pr{X=n} is then derived. However, it can be applied only in cases where the sum of α and β is large enough.

Finally, a numerical method for estimating Pr{X=n} is presented in Section IV. It can be seen that this is a powerful method when high accuracy is desired. However, it is also complicated and can be best used by means of a high speed digital computer.

As recommendations for further investigations, the following are considered as potential topics:

1. Determining the exact expression of Pr{X=n}. It is believed that such a formula can be obtained by advanced mathematical analysis.

2. Determining an adequate estimation procedure for the parameters involved in the prior distribution of Λ as specified in (2.1).

3. Continuing with an extension of this model to a bivariate or more general, k-variate model.

APPENDIX A

COMPUTERIZING THE CALCULATION OF P(n,s) AND DETAILED TREATMENT OF THE ERROR ESTIMATION

The following shows the development of the modified algorithm 2 in Section IV into details to computerize the calculation of the P(n,s) and the error involved u(n,s) as well as the estimated error $\varepsilon(n,s)$.

Suppose the values of α,β,b and P_0, d_0, S_0, B_0 and P_1, d_1, S_1, B_1 were already input. We now wish to compute P(n,s), u(n,s) and the corresponding $\varepsilon(n,s)$ for n = 0,1,...,N. Since the computation is done by an iterative method, the stopping criteria should be determined first.

Let r* be the maximum of the u(n,s) for n = 0.1...,N at the s^{th} iteration. The iteration is then stopped when $r^* \leq r$, where r is a small quantity determined in advance.

Step 1. Initialize all the P(n,s) to zero.
Set $E(s) = .5(1-e^{-2b})$
Set j = 0, where j is the subscript of the P_j.

Step 2. For j = 2,3,... and note that s is the actual value of j in this step, then
Compute P_j, S_j, B_j, d_j by (4.5), (4.9), (4.10), and (4.11)
Compute $t_k^{\,j}$.
Set $E(s) = E(s) - (d_j)^2\, S_j$

Step 3. For $n = 0,1,\ldots, N$ and for $j = 0,1,\ldots$

$$\text{Compute } T(j) = \frac{b^n\Gamma(\alpha+\beta)}{n!\Gamma(\alpha)} \sum_{k=0}^{j} t_k{}^j b^k \frac{\Gamma(\alpha+n+k)}{\Gamma(\alpha+\beta+n+k)}$$

(a.1) Set $P(n,s) = P(n,s) + T(j)$

Compute $u(n,s)$ by (4.15).

Compute $\varepsilon(n,s)$ by (4.18) for $j > 0$

First by solving for λ_1 in:

$$\text{(a.2)} \qquad f(\lambda) = p_s(\lambda) - \frac{P(n,s)}{V} = 0$$

As mentioned previously (a.2) is solved using Newton iterative method [4]. The idea is to generate the sequence $\{\lambda^{(i)}\}$ where $i = 0,1,\ldots$ by the iterative formula:

$$\text{(a.3)} \qquad \lambda^{(m+1)} = \lambda^{(m)} - \frac{f(\lambda^{(m)})}{f'(\lambda^{(m)})}$$

starting from $\lambda^{(0)}$ as defined in (4.21).

Furthermore, if the sequence so generated converges to some point ξ, i.e., for some inter L, we have

$$\text{(a.4)} \qquad \lambda^{(L+1)} = \lambda^{(L)} = \xi$$

so that (a.3) now becomes

$$\xi = \xi - \frac{f(\xi)}{f'(\xi)}$$

then it is immediate that $f(\xi) = 0$, and that ξ is the desired λ_1.

Preliminary, by (a.3), $f(\lambda)$ should be continuously differentiable and its derivative with respect to λ should not vanish at $\xi = \lambda_1$. However, referring to the analysis done in Section IV, it can be seen that $f(\xi)$ satisfy both of the above conditions.

On the other hand, starting with $\lambda^{(0)}$ as found in (4.21) the sequence $\{\lambda^{(0)}\}$ will converge to λ_1 quadratically. The proof is as follows.

Consider,

$$h(\lambda) = \lambda - \frac{f(\lambda)}{f'(\lambda)} \tag{a.5}$$

Taking the derivative of (a.5) with respect to λ,

$$\frac{d}{d\lambda} h(\lambda) = \frac{(p_s(\lambda) - \frac{P(n,s)}{V})\ f''(\lambda)}{[f'(\lambda)]^2} \tag{a.6}$$

Now, by noting that, in the vicinity of λ_1, (a.6) is small with respect to 1, and it vanishes at λ_1, then making use of the discussion in page 57 of the reference, it follows that the iteration converges quadratically to λ_1.

But, the integer L as in (a.4) is not known, and usually is estimated by L* in advance. Hence, λ_1 may be found exactly or with some tolerance within L* Newton iterations, if $\lambda^{(0)}$ is close enough to λ_1.

Based on the above argument, we may then stop the searching process for λ_1 if for some m the following conditions are met.

(a.7) $$|\lambda^{(m+1)} - \lambda^{(m)}| \leq \phi$$

(a.8) $$|f(\lambda^{(m)})| \leq \theta$$

It is noted that $m \leq L^*$ and that ϕ, θ are small positive quantities.

The above conditions are checked by associating them with the binary function Flag defined as below

$$\text{Flag} = \begin{cases} 0 & \text{if (a.7) or (a.8) are satisfied} \\ 1 & \text{otherwise} \end{cases}$$

Then during the course of searching for a λ_1, if Flag comes out with the value 1, either one of the following corrections have to be made.

1. Increasing L^*
2. Substituting $\lambda^{(0)}$ by $\lambda^{(L^*)}$.

The corrections may be repeated until Flag = 0.

Summarizing the above analysis in the algorithmic form we have:

For $j = 1,2,\ldots$

Compute $\lambda^{(0)}$ by (4.21)

Solve for λ_1 in (a.2)

Find the value of Flag, make the correction if Flag = 1 until Flag = 0, if $\lambda^{(0)}$ is close enough to λ_1.

Compute $\varepsilon(n,s)$

Step 4. Check if $u(n,s)$ satisfied the stopping criteria for $n = 0,1,\ldots,N$.

If $r^* \leq r$ Stop

If $r^* > r$ Update j by j + 1 then go to Step 2.

The application of the above results is seen with the illustration in Section V.

APPENDIX B

TABLES

The contents of this appendix are for the purpose of illustrating the convergence of P(n,s), for n = 0,1,...,6 as in the example presented in Section V. This is also known as a direct application of the results developed in Section IV and in Appendix A.

The values of P(n,s), u(n,s), ε(n,s), λ_1, f(λ_1) and Flag at the j th iteration (j=2,3,...) are tabulated in tables numbered from IX to XVIII.

The following abbreviations are used.

P(X=n):	P(n,s), n = 0,1,...,6.
BOUND:	The bound of the error, i.e., u(n,s).
EST. ERROR:	The estimated error, i.e., ε(n,s).
XO:	The value of λ_1.
FXO:	f(λ_1).
IFLAG:	Flag.

It is noted that the accuracy desired is achieved at 11 iterations, with r = .156 10^{-5}, the tolerance error in λ_1 and in f(λ_1) less than 10^{-15} ($\phi = \theta = 10^{-15}$), and the maximum number of Newton iterations allowed (L*) at 15.

The generated tables shown in this appendix and all computer work presented in this thesis were done utilizing the

IBM 360/67 computer at the Computer Facility of The Naval Postgraduate School, and programmed in Fortran-Formac and in Fortran G using double precision.

TABLE IX

ITERATION # 2

PR(X=0),BOUND	0.37555234D 00	0.23013736D 00	
EST. ERROR		-0.51095633D-01	
XC,FXC,IFLAG	0.11256032D 01	0.13877788D-16	0
PR(X=1),BOUND	0.37899285D 00	0.24517190D 00	
EST. ERROR		-0.12110029D 00	
XC,FXC,IFLAG	0.14735313D 01	0.0	0
PR(X=2),BOUND	0.25130315D 00	0.17647110D 00	
EST. ERROR		-0.11076349D 00	
XC,FXC,IFLAG	0.17750992D 01	0.13877788D-16	0
PR(X=3),BOUND	0.13423654D 00	0.10355327D 00	
EST. ERROR		-0.70367241D-01	
XC,FXC,IFLAG	0.20389942D 01	0.13877788D-16	0
PR(X=4),BOUND	0.61922828D-01	0.52781511D-01	
EST. ERROR		-0.36129014D-01	
XC,FXC,IFLAG	0.22718605D 01	0.13877788D-16	0
PR(X=5),BOUND	0.25508930D-01	0.24097998D-01	
EST. ERROR		-0.15980409D-01	
XC,FXC,IFLAG	0.24788669D 01	0.19081958D-16	0
PR(X=6),BOUND	0.95677015D-02	0.10034512D-01	
EST. ERROR		-0.62966740D-02	
XC,FXC,IFLAG	0.26640943D 01	0.29490299D-16	0

R* IS 0.245171903D 00

GREATER THAN R = 0.15600000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,

ITERATION CONTINUES..

TABLE X

ITERATION # 3

PR(X=0),BOUND	0.41672315D 00	0.99030114D-01	
EST. ERROR		-0.71339824D-01	
XC,FXC,IFLAG	0.10631004D 01	0.40766002D-16	0
PR(X=1),BOUND	0.36744446D 00	0.10549961D 00	
EST. ERROR		-0.88202120D-01	
XC,FXC,IFLAG	0.13939943D 01	0.52909066D-16	0
PR(X=2),BOUND	0.21239926D 00	0.75937054D-01	
EST. ERROR		-0.58050477D-01	
XC,FXC,IFLAG	0.16813740D 01	0.50306981D-16	0
PR(X=3),BOUND	0.98482140D-01	0.44559877D-01	
EST. ERROR		-0.27475991D-01	
XC,FXC,IFLAG	0.19330664D 01	-0.13877788D-16	0
PR(X=4),BOUND	0.39169598D-01	0.22712345D-01	
EST. ERROR		-0.10180979D-01	
XC,FXC,IFLAG	0.21550919D 01	0.13877788D-16	0
PR(X=5),BOUND	0.13775805D-01	0.10369579D-01	
EST. ERROR		-0.29600250D-02	
XC,FXC,IFLAG	0.23521502D 01	0.27755576D-16	0
PR(X=6),BOUND	0.43490496D-02	0.43179379D-02	
EST. ERROR		-0.60097585D-03	
XC,FXC,IFLAG	0.25279564D 01	0.0	0

R* IS 0.105499609D 00

GREATER THAN R = 0.156000000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,
ITERATION CONTINUES..

TABLE XI

ITERATION # 4

PR(X=0),BOUND	0.39622318D 00	0.33768592D-01	
EST. ERROR		-0.29617992D-01	
XO,FXO,IFLAG	0.10034698D 01	0.30357661D-16	0
PR(X=1),BOUND	0.32489736D 00	0.35974645D-01	
EST. ERROR		-0.23314887D-01	
XO,FXO,IFLAG	0.13170308D 01	0.86736174D-17	0
PR(X=2),BOUND	0.17694467D 00	0.25894016D-01	
EST. ERROR		-0.77105091D-02	
XO,FXO,IFLAG	0.15893056D 01	0.27755576D-16	0
PR(X=3),BOUND	0.78528452D-01	0.15194613D-01	
EST. ERROR		0.39970397D-03	
XO,FXO,IFLAG	0.18272949D 01	0.97144515D-16	0
PR(X=4),BOUND	0.30485588D-01	0.77447542D-02	
EST. ERROR		0.21573034D-02	
XO,FXO,IFLAG	0.20363681D 01	0.28622937D-16	0
PR(X=5),BOUND	0.10725709D-01	0.35359554D-02	
EST. ERROR		0.16097693D-02	
XO,FXO,IFLAG	0.22206769D 01	0.39898640D-16	0
PR(X=6),BOUND	0.34952633D-02	0.14723873D-02	
EST. ERROR		0.83556278D-03	
XO,FXO,IFLAG	0.23834402D 01	0.19949320D-16	0

R* IS 0.359746452D-01

GREATER THAN R = 0.156000000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,

ITERATION CONTINUES..

TABLE XII

ITERATION # 5

PR(X=0),BOUND	0.38539646D 00	0.94896277D-02	
EST. ERROR		-0.57946093D-02	
XC,FXC,IFLAG	0.96863234D 00	-0.52041704D-17	0
PR(X=1),BOUND	0.31364717D 00	0.10109571D-01	
EST. ERROR		-0.15966919D-03	
XC,FXC,IFLAG	0.12783148D 01	0.19949320D-16	0
PR(X=2),BOUND	0.17234396D 00	0.72767196D-02	
EST. ERROR		0.33161124D-02	
XC,FXC,IFLAG	0.15520382D 01	0.43368087D-16	0
PR(X=3),BOUND	0.78355716D-01	0.42699803D-02	
EST. ERROR		0.30133058D-02	
XC,FXC,IFLAG	0.17968383D 01	0.31225023D-16	0
PR(X=4),BOUND	0.31570099D-01	0.21764258D-02	
EST. ERROR		0.16656015D-02	
XC,FXC,IFLAG	0.20183706D 01	0.54914840D-16	0
PR(X=5),BOUND	0.11640117D-01	0.99367189D-03	
EST. ERROR		0.68832003D-03	
XC,FXC,IFLAG	0.22212479D 01	0.28622937D-16	0
PR(X=6),BOUND	0.39968813D-02	0.41376932D-03	
EST. ERROR		0.22363482D-03	
XC,FXC,IFLAG	0.24092410D 01	0.29490299D-16	0

R* IS 0.10109571 4D-01

GREATER THAN R = 0.15600000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,
ITERATION CONTINUES..

TABLE XIII

ITERATION # 6

PR(X=0),BOUND	0.38342695D 00	0.22621386D-02	
EST. ERROR		0.25157580D-06	
XC,FXC,IFLAG	0.95860549D 00	0.19949320D-16	0
PR(X=1),BOUND	0.31324038D 00	0.24099208D-02	
EST. ERROR		0.15980352D-02	
XC,FXC,IFLAG	0.12740148D 01	-0.30303451D-16	0
PR(X=2),BOUND	0.17335804D 00	0.17346253D-02	
EST. ERROR		0.14157378D-02	
XC,FXC,IFLAG	0.15570965D 01	0.52583805D-16	0
PR(X=3),BOUND	0.79420638D-01	0.10178784D-02	
EST. ERROR		0.61894872D-03	
XC,FXC,IFLAG	0.18133116D 01	0.46837534D-16	0
PR(X=4),BOUND	0.32183523D-01	0.51881663D-03	
EST. ERROR		0.12907845D-03	
XC,FXC,IFLAG	0.20465379D 01	0.86736174D-18	0
PR(X=5),BOUND	0.11892659D-01	0.23687162D-03	
EST. ERROR		-0.26717150D-04	
XC,FXC,IFLAG	0.22594841D 01	-0.32959746D-16	0
PR(X=6),BOUND	0.40747492D-02	0.98634378D-04	
EST. ERROR		-0.39266563D-04	
XC,FXC,IFLAG	0.24540726D 01	0.14918622D-15	0

R* IS 0.240992081D-02

GREATER THAN R = 0.156000000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,

ITERATION CONTINUES..

TABLE XIV

ITERATION # 7

PR(X=0),BOUND	0.38328972D 00	0.46752892D-03	
EST. ERROR		0.28859940D-03	
XC,FXC,IFLAG	0.95821145D 00	-0.38489177D-17	0
PR(X=1),BOUND	0.31349161D 00	0.49807191D-03	
EST. ERROR		0.40242683D-03	
XC,FXC,IFLAG	0.12770189D 01	0.35832882D-16	0
PR(X=2),BOUND	0.17364839D 00	0.35850477D-03	
EST. ERROR		0.12492369D-03	
XC,FXC,IFLAG	0.15628372D 01	0.25424541D-16	0
PR(X=3),BOUND	0.79550689D-01	0.21037066D-03	
EST. ERROR		-0.45798214D-04	
XC,FXC,IFLAG	0.18200144D 01	0.10842022D-17	0
PR(X=4),BOUND	0.32201490D-01	0.10722676D-03	
EST. ERROR		-0.64428950D-04	
XC,FXC,IFLAG	0.20519853D 01	0.24123498D-16	0
PR(X=5),BOUND	0.11876448D-01	0.48955592D-04	
EST. ERROR		-0.36073577D-04	
XC,FXC,IFLAG	0.22616411D 01	0.99628015D-16	0
PR(X=6),BOUND	0.40594618D-02	0.20385323D-04	
EST. ERROR		-0.13709141D-04	
XC,FXC,IFLAG	0.24515309D 01	0.33556057D-15	0

R* IS 0.498071909D-03

GREATER THAN R = 0.156000000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,
ITERATION CONTINUES..

TABLE XV

ITERATION # 6

PR(X=0),BOUND	0.38329649D 00	0.85208736D-04	
EST. ERROR		0.72481097D-04	
XC,FXC,IFLAG	0.95875737D 00	0.45434847D-16	0
PR(X=1),BOUND	0.31353580D 00	0.90775300D-04	
EST. ERROR		0.25971392D-04	
XC,FXC,IFLAG	0.12780779D 01	0.95409791D-16	0
PR(X=2),BOUND	0.17366418D 00	0.65338715D-04	
EST. ERROR		-0.30455858D-04	
XC,FXC,IFLAG	0.15636408D 01	-0.46112474D-16	0
PR(X=3),BOUND	0.79541480D-01	0.38340769D-04	
EST. ERROR		-0.29830734D-04	
XC,FXC,IFLAG	0.18199294D 01	0.31015297D-15	0
PR(X=4),BOUND	0.32189502D-01	0.19542441D-04	
EST. ERROR		-0.12633620D-04	
XC,FXC,IFLAG	0.20507474D 01	0.38125631D-15	0
PR(X=5),BOUND	0.11869791D-01	0.89223231D-05	
EST. ERROR		-0.25225713D-05	
XC,FXC,IFLAG	0.22593723D 01	0.14793939D-15	0
PR(X=6),BOUND	0.40570559D-02	0.37152944D-05	
EST. ERROR		0.43041158D-06	
XC,FXC,IFLAG	0.24486348D 01	0.63967928D-17	0

R* IS 0.907752997D-04

GREATER THAN R = 0.156000000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,
ITERATION CONTINUES..

TABLE XVI

ITERATION # 9

PR(X=C),BCUNC	0.38329881D C0	0.1388C549D-04	
EST. ERRCR		0.7730144lD-05	
XC,FXC,IFLAG	C.95892C26D CC	0.85685853D-17	C
PR(X=1),BCUNC	0.313538855D CC	0.14787346D-04	
EST. ERRCR		-C.68844768D-05	
XC,FXC,IFLAG	0.12781740D C1	0.72868550D-16	0
PR(X=2),BCUNC	0.17366227D C0	0.10643712D-04	
EST. ERRCR		-0.85494961D-05	
XC,FXC,IFLAG	0.15635257D C1	C.11780534D-16	0
PR(X=3),BCUNC	0.79539011D-C1	0.62457321D-05	
EST. ERRCR		-0.26186557D-05	
XC,FXC,IFLAG	0.18196182D C1	C.27816562D-16	0
PR(X=4),BCUNC	0.32188494D-C1	0.31834742D-05	
EST. ERRCR		C.53571488D-06	
XC,FXC,IFLAG	0.205036950 01	-0.81416807D-17	0
PR(X=5),BCUNC	0.11869734D-C1	0.14534513D-05	
EST. ERRCR		0.85567C10D-06	
XC,FXC,IFLAG	0.22590925D C1	C.76314280D-16	C
PR(X=6),BCUNC	0.40572299D-02	0.60522347D-06	
EST. ERRCR		0.43849325D-C6	
XC,FXC,IFLAG	0.24485899D 01	-C.16399606D-15	C

R* IS 0.147873455D-04

GREATER THAN R = 0.15600C0CCD-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,
ITERATION CONTINUES..

TABLE XVII

ITERATION # 10

PR(X=0),BOUND	0.38329906D 00	0.20433230D-05	
EST. ERROR		-0.15910869D-06	
XC,FXC,IFLAG	0.95894018D 00	-0.24387561D-16	0
PR(X=1),BOUND	0.31353853D 00	0.21768104D-05	
EST. ERROR		-0.17961116D-05	
XC,FXC,IFLAG	0.12781578D 01	0.11269329D-15	0
PR(X=2),BOUND	0.17366198D 00	0.15668358D-05	
EST. ERROR		-0.50556254D-06	
XC,FXC,IFLAG	0.15634810D 01	-0.28111541D-16	0
PR(X=3),BOUND	0.79538919D-01	0.91941953D-06	
EST. ERROR		0.40043990D-06	
XC,FXC,IFLAG	0.18195814D 01	0.61765007D-16	0
PR(X=4),BOUND	0.32188554D-01	0.46863175D-06	
EST. ERROR		0.35498815D-06	
XC,FXC,IFLAG	0.20503732D 01	-0.17616368D-15	0
PR(X=5),BOUND	0.11869801D-01	0.21395914D-06	
EST. ERROR		0.12640418D-06	
XC,FXC,IFLAG	0.22591483D 01	0.26446274D-16	0
PR(X=6),BOUND	0.40572610D-02	0.89093523D-07	
EST. ERROR		0.15218146D-07	
XC,FXC,IFLAG	0.24486866D 01	0.72410941D-16	0

R* IS 0.217681035D-05

GREATER THAN R = 0.156000000D-05

ACCURACY DESIRED HAS NOT BEEN ACHIEVED,

ITERATION CONTINUES..

TABLE XVIII

ITERATION # 11

PR(X=0),BOUND	0.38329908D 00	0.27432726D-06	
EST. ERROR		-0.18032066D-06	
XO,FXO,IFLAG	0.95894018D 00	-0.74565154D-17	0
PR(X=1),BOUND	0.31353852D 00	0.29224867D-06	
EST. ERROR		-0.14481587D-06	
XO,FXO,IFLAG	0.12781526D 01	0.16005895D-15	0
PR(X=2),BOUND	0.17366197D 00	0.21035626D-06	
EST. ERROR		0.10193154D-06	
XO,FXO,IFLAG	0.15634776D 01	0.19095815D-16	0
PR(X=3),BOUND	0.79538928D-01	0.12343709D-06	
EST. ERROR		0.94961000D-07	
XO,FXO,IFLAG	0.18195852D 01	0.37754707D-16	0
PR(X=4),BOUND	0.32188563D-01	0.62916370D-07	
EST. ERROR		0.20627421D-07	
XO,FXO,IFLAG	0.20503833D 01	-0.54260084D-16	0
PR(X=5),BOUND	0.11869803D-01	0.28725182D-07	
EST. ERROR		-0.81489324D-08	
XO,FXO,IFLAG	0.22591594D 01	0.14779243D-16	0
PR(X=6),BOUND	0.40572604D-02	0.11961292D-07	
EST. ERROR		-0.78825133D-08	
XO,FXO,IFLAG	0.24486924D 01	-0.10202504D-15	0

ACCURACY DESIRED ACHIEVED AT 11TH ITERATION

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A discrete distribution arises from a Poisson distribution with parameter λ, when the distribution of λ itself is of the form $c\lambda^{\alpha-1}(b-\lambda)^{\beta-1}$ $\lambda \varepsilon [0,b]$, where c is a scaling factor and α, β, b are strictly positive parameters. However, the functional form of the resulting unconditional distribution is not particularly

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tractable, hence the study of the statistical properties of the unconditional distribution is limited in the study of its mean and its variance.

In regards to the modelling of a real situation, an estimation procedure of the parameters involved in $c\lambda^{\alpha-1}(b-\lambda)^{\beta-1}$ is discussed and a closed form of the probability distribution is derived. In addition, when accuracy is desired a numerical analysis of the probability distribution is also presented. The development of the results is continued in Appendix A, as a preparation in computerizing the calculation.

Finally, an application to real data is discussed for the purpose of illustrating the model.

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On the Poisson variate with beta prior distribution and a numerical method in estimating the resulting unconditional distribution.